Envelope and phase statistics of Cauchy quadratures

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Non-Gaussian quadratures, especially Cauchy quadratures, are widely used to model impulsive noise environments. The envelope and phase statistics of Cauchy quadratures are investigated. The probability density functions and cumulative distribution functions of the envelope and phase of Cauchy quadratures are given in closed-form expression. The characteristics of the derived distributions are also discussed.

Introduction: Non-Gaussian quadrature models find widespread applications in impulsive noise environments such as wireless communication, signal detection, and signal processing [1–2]. Among them, Cauchy quadrature model is one of the most commonly encountered non-Gaussian models in communication studies. The Cauchy probability model is widely used to model strong impulsive additive noise such as electromagnetic interference [3] and under-ice noise [4].

As system designers strive to attain the maximal possible performance from communications systems, and as applications of data communications spread into new and foreign transmission environments, it is now important to design for non-Gaussian quadratures in addition to the ubiquitous Gaussian quadratures. Although the Gaussian quadratures are very well understood and modelled, there is less understanding of non-Gaussian quadratures, and in particular, of the Cauchy quadratures, despite their widespread application.

Therefore, the stimulus of obtaining optimal performance from communication systems encourages researchers to study the statistics of non-Gaussian quadratures, and in particular of Cauchy quadratures. The literature reports the envelope and phase statistics of non-Gaussian quadratures based on different assumptions. Roberts [5] derived the envelope statistics of Cauchy quadrature model based on the assumption that Cauchy noise is ergodic and circularly symmetric. Dias and Yacoub [6] reported the $\kappa-\mu$ joint phase–envelope distribution based on the independence of quadrature components. Yacoub [7] adopted the same assumption and obtained a new Nakagami-m phase–envelope distribution. In this Letter, we derive the envelope and phase statistics for independent Cauchy quadratures.

Cauchy noise: Let $N = X + \mathrm{j}Y$ denote the Cauchy quadrature model, where X and Y are independent Cauchy quadratures. We can also rewrite N as $N = R\exp(\mathrm{j}\Theta)$, where R and Θ are the envelope and phase of N and are given by

$$R = \sqrt{X^2 + Y^2}$$
$$\Theta = \tan^{-1}\left(\frac{Y}{X}\right) \tag{1}$$

The probability density function (PDF) of the independent Cauchy quadratures is given by

$$f_Z(z) = \frac{b}{\pi\left(b^2 + z^2\right)} \tag{2}$$

where $Z = X$ or $Z = Y$, b is a scaling parameter. Since X and Y are independent random variables, the joint two-dimensional PDF of them is

$$f_{XY}(x, y) = \frac{b^2}{\pi^2\left(b^2 + y^2\right)\left(b^2 + x^2\right)} \tag{3}$$

Using the transformations in (1), $f_{R\Theta}(r, \theta) = |J| f_{XY}(x, y)$, where $|J|$ is the Jacobian determinant of the transformations in (1). Note that $|J| = r$. Thus, the joint envelope–phase distribution of N is given by

$$f_{R\Theta}(r, \theta) = \frac{b^2 r}{\pi^2\left(b^2 + r^2\sin^2\theta\right)\left(b^2 + r^2\cos^2\theta\right)} \tag{4}$$

Doing a double integration of the joint PDF of R and Θ gives the joint cumulative distribution function (CDF) of R and Θ

$$F_{R\Theta}(r, \theta) = \int_0^{\theta} \frac{1}{2\pi^2\cos 2\phi} \ln\left(\frac{b^2 + r^2\cos^2\phi}{b^2 + r^2\sin^2\phi}\right) \mathrm{d}\phi \tag{5}$$

Although the closed-form expression of (5) cannot be derived, it can be numerically computed. The marginal statistics of R and Θ are derived by integrating (4) on r and θ, respectively. To derive the marginal PDF of R, we first rewrite (4) as

$$f_{R\Theta}(r, \theta) = \frac{b^2 r}{\pi^2\left[b^2\left(b^2 + r^2\right)\right]} \frac{1}{1 + c\sin^2 2\theta} \tag{6}$$

where $c = r^4/[4b^2(b^2 + r^2)]$. Integrating (6) from $\theta = 0$ to 2π, the marginal PDF of R is obtained

$$f_R(r) = \frac{4br}{\pi\sqrt{\left(b^2 + r^2\right)\left(2b^2 + r^2\right)}}, \quad r \geq 0 \tag{7}$$

Furthermore, we derive the marginal CDF of R as

$$F_R(r) = \frac{4}{\pi}\arctan\left(\frac{\sqrt{b^2 + r^2}}{b}\right) - 1, \quad r \geq 0 \tag{8}$$

Similarly, to obtain the marginal PDF of Θ, we first need to rewrite (6) as two separated parts, i.e.

$$f_{R\Theta}(r, \theta) = \frac{1}{\pi^2\cos 2\theta}\left[\cos^2\theta \frac{r}{b^2 + r^2\cos^2\theta} - \sin^2\theta \frac{r}{b^2 + r^2\sin^2\theta}\right] \tag{9}$$

Then integrating (9) with respect to r from 0 to ∞, one gets the PDF

$$f_\Theta(\theta) = \frac{\ln\left(\tan^2\theta\right)}{2\pi^2\left(\tan^2\theta - 1\right)\cos^2\theta}, \quad 0 \leq \theta \leq 2\pi \tag{10}$$

Finally, integration of (10) gives the CDF of Θ, i.e.

$$F_\Theta(\theta) = \frac{Li_2(1 - p_\theta\cot\theta) + Li_2(-p_\theta\cot\theta)}{2p_\theta\pi^2} + \frac{\ln(p_\theta\cot\theta)\ln(1 + p_\theta\cot\theta)}{2p_\theta\pi^2} + s(\theta), \quad 0 \leq \theta \leq 2\pi \tag{11}$$

where $Li_2(\cdot)$ is a polylogarithm function [8], $p_\theta = \mathrm{sgn}(\cot\theta)$, and $s(\theta)$ is a staircase function given by

$$s(\theta) = \begin{cases} \frac{1}{6}, & 0 \leq \theta \leq \frac{\pi}{2} \\ \frac{1}{3}, & \frac{\pi}{2} < \theta \leq \pi \\ \frac{2}{3}, & \pi < \theta \leq \frac{3\pi}{2} \\ \frac{5}{6}, & \frac{3\pi}{2} < \theta \leq 2\pi \end{cases} \tag{12}$$

Discussion: The envelope distribution is an unimodal curve. The peak of $f_R(r)$ occurs at $r_1 = b\sqrt{\sqrt{17} - 1}/2$ and its value is

$$f_R(r_1) = \frac{16\sqrt{\sqrt{17} - 1}}{\pi b\left(\sqrt{17} + 7\right)\sqrt{\sqrt{17} + 3}} \tag{13}$$

The phase PDF tends to be infinite at $\theta = \pi/2$, π, $3\pi/3$, and 2π because the denominator approaches 0 at these locations of θ. The reason is that Cauchy distribution is a heavy-tailed distribution with only 50% probability falls in the interval $[-b, b]$. When the quadrature components are distributed along the x and y axes, more energy are gathered at $\theta = \pi/2$, π, $3\pi/3$, and 2π. When $\theta = \pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$, both the numerator and the denominator tend to be 0. In this case, the limit of $f_\Theta(\theta)$ towards $\pi/4$ is $1/\pi^2$, i.e.

$$\lim_{\theta \to \pi/4} \frac{\ln(\tan^2\theta)}{2\pi^2(\tan^2\theta - 1)\cos^2\theta} = \frac{1}{\pi^2} \tag{14}$$

For other locations of θ, we can get the same limit value.

Fig. 1 shows the PDF of the envelope of Cauchy noise for values of $b = 1$, 2, 4, and 8. The location of the mode of the envelope PDF approaches 0 with the decrease of b, while the height of the mode increases. This is also obvious from (13). The envelope PDF will be similar to an impulsive function with a much larger value of b because more probability is distributed around 0. Fig. 2 shows the corresponding phase PDF. Note from (10) and (11) that the phase PDF and CDF are independent of the parameter b, so each is represented by only one curve. Observe also that the Cauchy noise phase PDF has singularities at $\theta = 0$, $\pi/2$, π, and $3\pi/2$.

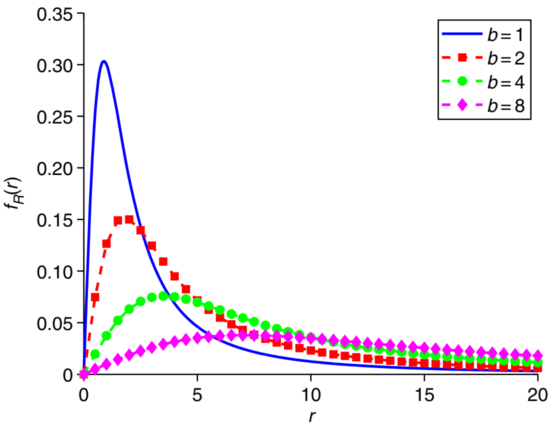

Fig. 1 *Envelope PDF*

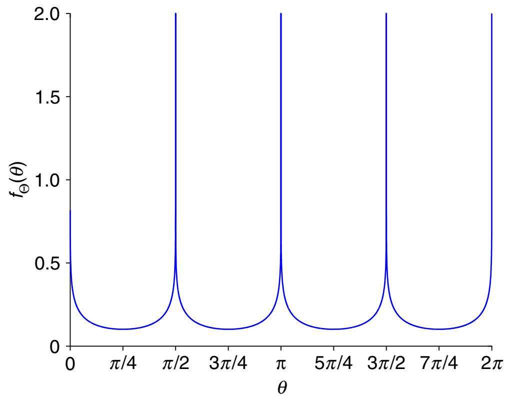

Fig. 2 *Phase PDF*

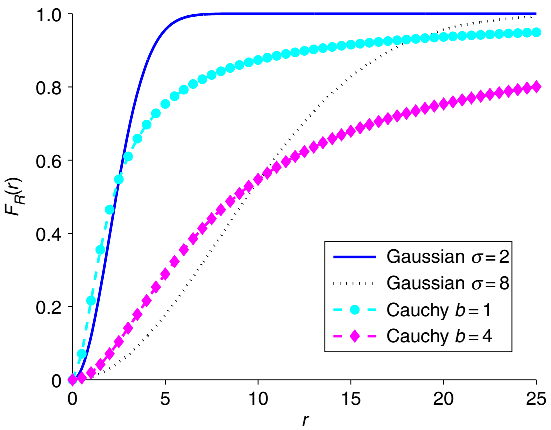

Fig. 3 *Comparison between envelope CDF of Gaussian and Cauchy quadratures*

Comparison with Gaussian quadratures: The envelope and phase PDFs of independent Gaussian quadratures are Rayleigh distribution with a parameter σ and uniform distribution over $[0, 2\pi]$. The phase PDF of Cauchy quadratures is quite different from the uniform one of Gaussian quadratures. However, the two envelope PDFs are similar in that they are both unimodal with one controlling parameter. The maximum value of the envelope PDF of Gaussian quadratures is $1/\sqrt{e}\sigma$, occurring at $r_2 = \sigma$. The peaks of the two envelope PDF both fall with larger controlling parameters, while their locations move towards ∞. The main difference is that the envelope PDF of Cauchy quadratures has a much heavier tail. Fig. 3 shows and compares the envelope CDFs of Cauchy quadratures and Gaussian quadratures for $b = 1$ and $b = 4$. σ is chosen so that the peaks of the envelope PDF of Gaussian and Cauchy quadratures are equal, namely

$$\frac{16\sqrt{\sqrt{17}-1}}{\pi b\left(\sqrt{17}+7\right)\sqrt{\sqrt{17}+3}} = \frac{1}{\sqrt{e}\sigma} \tag{16}$$

Solving the equation above results in $\sigma \approx 2b$. Under this constraint, the heavy-tail effect of Cauchy distribution is obvious from Fig. 3 where the CDF of Cauchy envelope approaches 1 slower than Gaussian envelope at the tail part.

Conclusion: In this Letter, we have derived the envelope and phase distributions of Cauchy quadratures. The statistical characteristics of the envelope and phase PDF are also discussed. The envelope PDF is a unimodal curve and tends to be an impulsive function with the decrease of b. The phase PDF is not uniform with the minimum values occurring at $\theta = \pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$. It has singularities at $\theta = 0$, $\pi/2$, π, and $3\pi/2$. We also compare the envelope and phase statistics of Cauchy quadratures with those of Gaussian quadratures. The Cauchy envelope has a much heavier tail than Gaussian envelope.

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One or more of the Figures in this Letter are available in colour online.

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